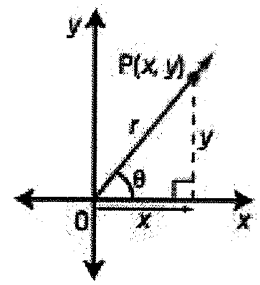
Answer Key

Math 20-1 Topic 2.2 Notes & Class Examples Trig Ratios of Any Angle

Finding the Trigonometric Ratios of Any Angle θ, where $0° \le \theta < 360°$

Suppose θ is any angle in standard position, and P(x, y) is any point on its terminal arm, at a distance r from the origin. Then, by the Pythagorean Theorem, $r = \sqrt{x^2 + y^2}$.



You can use a reference triangle to determine the three primary trigonometric ratios in terms of x, y, and r.

$\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$ $\quad \cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ $\quad \tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

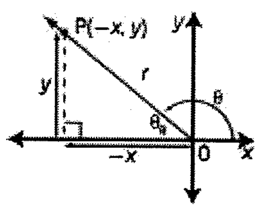
$\sin\theta = \dfrac{y}{r}$ $\quad \cos\theta = \dfrac{x}{r}$ $\quad \tan\theta = \dfrac{y}{x}$

The chart below summarizes the signs of the trigonometric ratios in each quadrant. In each, the horizontal and vertical lengths are considered as directed distances.

Quadrant II
$90° < \theta < 180°$

$\sin\theta = \frac{y}{r}$ $\quad \cos\theta = \frac{-x}{r}$ $\quad \tan\theta = \frac{y}{-x}$

$\sin\theta > 0$ $\quad \cos\theta < 0$ $\quad \tan\theta < 0$



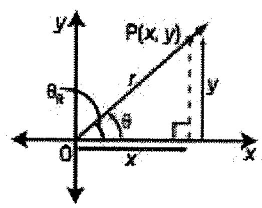
$\theta = 180° - \theta_R$

sin (pos)
cos (neg)
tan (neg)

Quadrant I
$0° < \theta < 90°$

$\sin\theta = \frac{y}{r}$ $\quad \cos\theta = \frac{x}{r}$ $\quad \tan\theta = \frac{y}{x}$

$\sin\theta > 0$ $\quad \cos\theta > 0$ $\quad \tan\theta > 0$



$\theta = \theta_R$

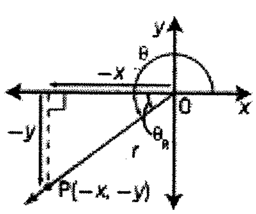
Why is r always positive?

sin (pos)
cos (pos)
tan (pos)

Quadrant III
$180° < \theta < 270°$

$\sin\theta = \frac{-y}{r}$ $\quad \cos\theta = \frac{-x}{r}$ $\quad \tan\theta = \frac{-y}{-x}$

$\sin\theta < 0$ $\quad \cos\theta < 0$ $\quad \tan\theta > 0$



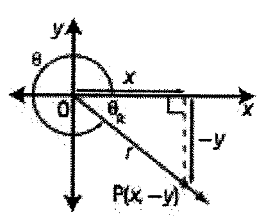
$\theta = 180° + \theta_R$

sin (neg)
cos (neg)
tan (pos)

Quadrant IV
$270° < \theta < 360°$

$\sin\theta = \frac{-y}{r}$ $\quad \cos\theta = \frac{x}{r}$ $\quad \tan\theta = \frac{-y}{x}$

$\sin\theta < 0$ $\quad \cos\theta > 0$ $\quad \tan\theta < 0$



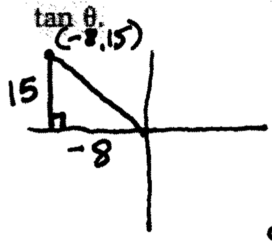
$\theta = 360° - \theta_R$

sin (neg)
cos (pos)
tan (neg)

Example 1

Write Trigonometric Ratios for Angles in Any Quadrant

The point P(−8, 15) lies on the terminal arm of an angle, θ, in standard position. Determine the exact trigonometric ratios for sin θ, cos θ, and tan θ.



$r = \sqrt{(-8)^2 + (15)^2}$

$r = \sqrt{289}$

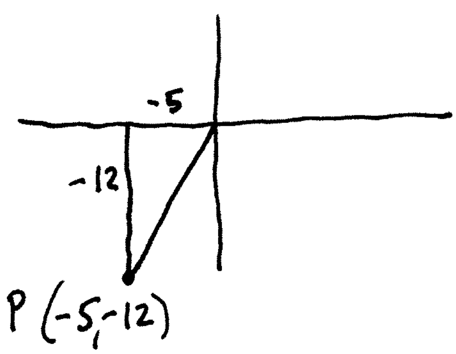
$r = 17$

$\sin\theta = \frac{y}{r}$ $\quad$ $\cos\theta = \frac{x}{r}$ $\quad$ $\tan\theta = \frac{y}{x}$

$\sin\theta = \frac{15}{17}$ $\quad$ $\cos\theta = \frac{-8}{17}$ $\quad$ $\tan\theta = -\frac{15}{8}$

Your Turn

The point P(−5, −12) lies on the terminal arm of an angle, θ, in standard position. Determine the exact trigonometric ratios for sin θ, cos θ, and tan θ.



$\cos\theta = \frac{x}{r} = \frac{-5}{13}$

$\sin\theta = \frac{y}{r} = \frac{-12}{13}$

$\tan\theta = \frac{y}{x} = \frac{12}{5}$

$r = \sqrt{x^2 + y^2}$

$r = \sqrt{(-5)^2 + (-12)^2}$

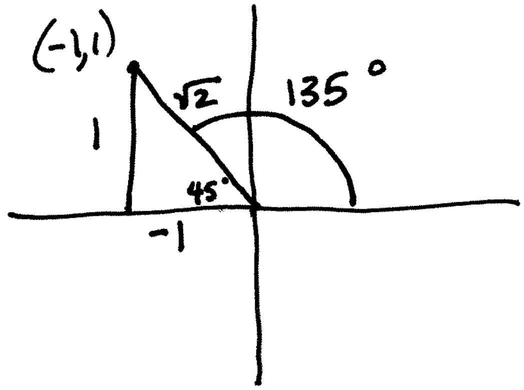
$r = \sqrt{169}$

$r = 13$

Example 2

Determine the Exact Value of a Trigonometric Ratio

Determine the exact value of cos 135°.



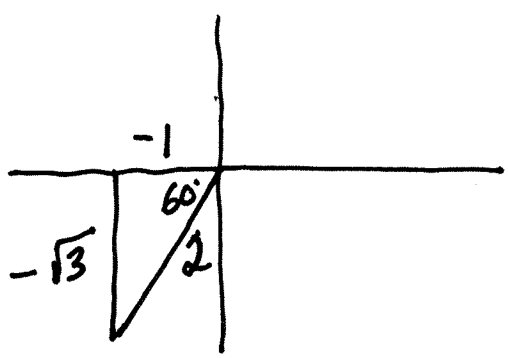
Reference θ

$$\cos 45 = \frac{adj}{hyp} = \frac{-1}{\sqrt{2}}$$

$$\cos 135° = \frac{-1}{\sqrt{2}}$$

Your Turn

Determine the exact value of sin 240°.



Reference θ

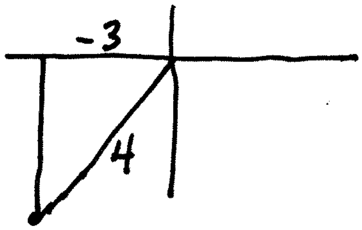
$$\sin 60 = \frac{opp}{hyp} = \frac{-\sqrt{3}}{2}$$

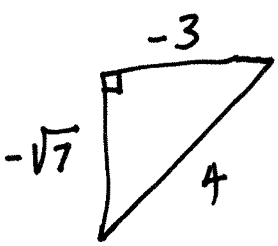
$$\sin 240° = \frac{-\sqrt{3}}{2}$$

Example 3

Determine Trigonometric Ratios

Suppose θ is an angle in standard position with terminal arm in quadrant III, and $\cos\theta = -\frac{3}{4}$. What are the exact values of $\sin\theta$ and $\tan\theta$?





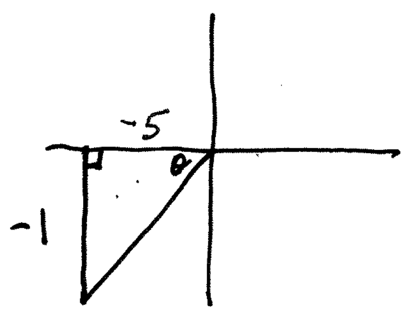
$$\cos\theta = \frac{adj}{hyp} = \frac{-3}{4}$$

$$a^2+b^2=c^2$$
$$(-3)^2+x^2=4^2$$
$$9+x^2=16$$
$$x^2=7$$
$$x=\sqrt{7}$$
$$\therefore\ y=-\sqrt{7}$$

$$\sin\theta = \frac{-\sqrt{7}}{4}$$
$$\tan\theta = \frac{\sqrt{7}}{\sqrt{3}}$$

Your Turn

Suppose θ is an angle in standard position with terminal arm in quadrant III, and $\tan\theta = \frac{1}{5}$. Determine the exact values of $\sin\theta$ and $\cos\theta$.



$$\sin\theta = \frac{-1}{\sqrt{26}}$$
$$\cos\theta = \frac{-5}{\sqrt{26}}$$

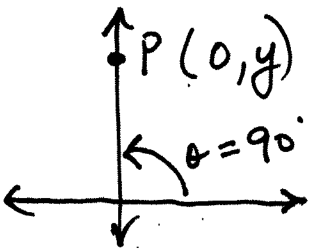
$$r=\sqrt{1^2+5^2}$$
$$r=\sqrt{1+25}$$
$$r=\sqrt{26}$$

Example 4

Determine Trigonometric Ratios of Quadrantal Angles

Determine the values of sin θ, cos θ, and tan θ when the terminal arm of quadrantal angle θ coincides with the positive y-axis, $\theta = 90°$.

quadrantal angles are only angles of 0, 90°, 180, 270° and 360°



$x = 0$
$y = y$
$y = r$

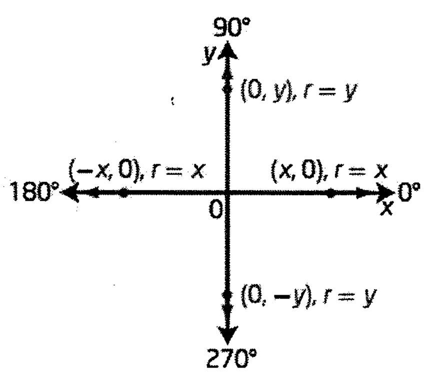
$$\sin 90° = \frac{y}{r} = \frac{y}{y} = 1$$

$$\cos 90° = \frac{x}{r} = \frac{0}{y} = 0$$

$$\tan 90° = \frac{y}{x} = \frac{y}{0} = \text{undefined}$$

Your Turn

Use the diagram to determine the values of sin θ, cos θ, and tan θ for quadrantal angles of 0°, 180°, and 270°. Organize your answers in a table as shown below.



	0°	90°	180°	270°
sin θ	0	1	0	−1
cos θ	1	0	−1	0
tan θ	0	undefined	0	undef.

Solving for Angles Given Their Sine, Cosine, or Tangent

Step 1 Determine which quadrants the solution(s) will be in by looking at the sign (+ or −) of the given ratio.

Step 2 Solve for the reference angle.

Why are the trigonometric ratios for the reference angle always positive?

Step 3 Sketch the reference angle in the appropriate quadrant. Use the diagram to determine the measure of the related angle in standard position.

Example 5

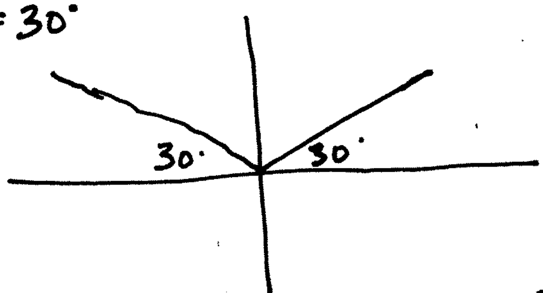
Solve for an Angle Given Its Exact Sine, Cosine, or Tangent Value

Solve for θ.

a) $\sin\theta = 0.5$, $0° \le \theta < 360°$

sin θ is pos in Quad 1 and 2

$\theta = 30°$

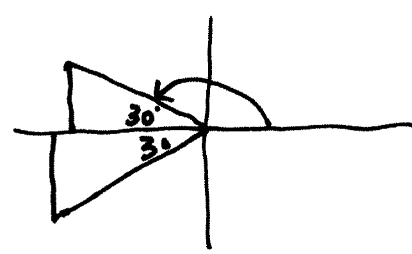


$\theta = 30°$ or $\theta = 150°$

b) $\cos\theta = -\frac{\sqrt{3}}{2}$, $0° \le \theta < 180°$

cos θ is neg in quad 2 and 3

and $\cos^{-1}\left(+\frac{\sqrt{3}}{2}\right) = 30°$



$\theta = 150°$

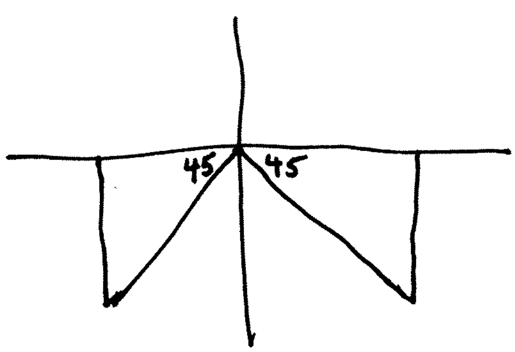
or

$\theta = 210°$

but our restriction of $0° \le \theta \le 180°$ means $\theta = 150°$

Your Turn

Solve $\sin\theta = -\frac{1}{\sqrt{2}}$, $0° \le \theta < 360°$.



sin θ is neg in quad 3 + 4

$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$

$\theta = 180 + 45 = 225°$

or

$\theta = 360 - 45 = 315°$

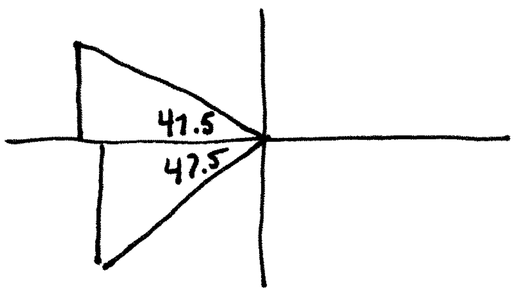
Example 6

Solve for an Angle Given Its Approximate Sine, Cosine, or Tangent Value

Given $\cos \theta = -0.6753$, where $0° \leq \theta < 360°$, determine the measure of θ, to the nearest tenth of a degree.

$\cos \theta$ is neg in quadrants 2 and 3

and $\cos^{-1}(0.6753) = 47.5°$ (This is our reference angle).



$\theta = 180 - 47.5° = 132.5°$

and

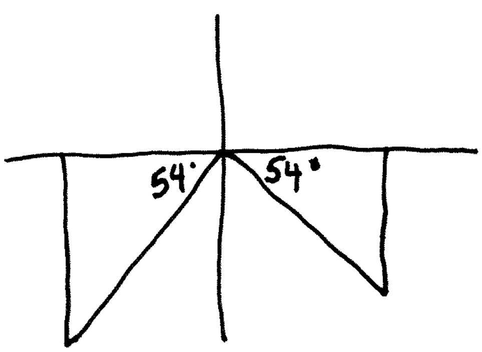
$\theta = 180 + 47.5 = 227.5°$

Your Turn

Determine the measure of θ, to the nearest degree, given $\sin \theta = -0.8090$, where $0° \leq \theta < 360°$.

$\sin \theta$ is neg in quadrants 3 and 4

and $\sin^{-1}(0.8090)$ is $54°$ (Reference angle - $54°$)



$\theta = 180 + 54° = 234°$

and

$\theta = 360 - 54 = 306°$

Key Ideas

- The primary trigonometric ratios for an angle, θ, in standard position that has a point $P(x, y)$ on its terminal arm are $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, and $\tan\theta = \frac{y}{x}$, where $r = \sqrt{x^2 + y^2}$.
- The table show the signs of the primary trigonometric ratios for an angle, θ, in standard position with the terminal arm in the given quadrant.

	Quadrant			
Ratio	**I**	**II**	**III**	**IV**
$\sin\theta$	+	+	−	−
$\cos\theta$	+	−	−	+
$\tan\theta$	+	−	+	−

- If the terminal arm of an angle, θ, in standard position lies on one of the axes, θ is called a quadrantal angle. The quadrantal angles are $0°$, $90°$, $180°$, $270°$, and $360°$, $0° \le \theta \le 360°$.