Math 20 – 1 Pre-Calculus 11 Topic 1.1 Arithmetic Sequences

Get Started

When the numbers on these plates are arranged in order, the differences between each number and the previous number are the same.

What are the missing numbers?

$\boxed{11}$ $\boxed{19}$ $\boxed{27}$ $\boxed{35}$

$+d$ $+d$ $+d$

$+8$ $+8$

$11 + 3d = 35$
$3d = 24$
$d = 8$

Construct Understanding

> Saket took guitar lessons.
>
> The first lesson cost \$75 and included the guitar rental for the period of the lessons.
>
> The total cost for 10 lessons was \$300.
>
> Suppose the lessons continued.
>
> What would be the total cost of 15 lessons?
>
> 1 lesson at 75
> 14 @ 25 = 425

$\boxed{75}$

1	75
10	300
15	?

Cost of 9 lessons = total cost of 10 − cost of 1st
= 300 − 75
= 225

Cost of 1 lesson $= \frac{225}{9} = 25$

In an **arithmetic sequence,** the difference between consecutive **terms** is constant. This constant value is called the **common difference.**

This is an arithmetic sequence:
4, 7, 10, 13, 16, 19, . . .
The first term of this sequence is: $t_1 = 4$
The second term is: $t_2 = 7$

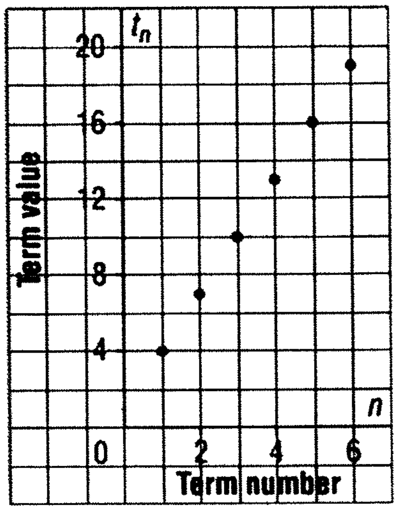
Let d represent the common difference. For the sequence above:

$d = t_2 - t_1$ and $d = t_3 - t_2$ and $d = t_4 - t_3$ and so on
$= 7 - 4$ $= 10 - 7$ $= 13 - 10$
$= 3$ $= 3$ $= 3$

$$4 \xrightarrow{+3} 7 \xrightarrow{+3} 10 \xrightarrow{+3} 13 \xrightarrow{+3} 16 \xrightarrow{+3} 19, \ldots$$

The dots indicate that the sequence continues forever; it is an **infinite arithmetic sequence**.
To graph this arithmetic sequence, plot the term value, t_n, against the term number, n.

Graph of an Arithmetic Sequence



The graph represents a linear function because the points lie on a straight line. A line through the points on the graph has slope 3, which is the common difference of the sequence.

In an arithmetic sequence, the common difference can be any real number.

Here are some other examples of arithmetic sequences.

- This is an *increasing* arithmetic sequence because d is positive and the terms are increasing:
  $\frac{1}{2}, \frac{3}{4}, 1, 1\frac{1}{4}, \ldots$; with $d = \frac{1}{4}$
- This is a *decreasing* arithmetic sequence because d is negative and the terms are decreasing:
  $5, -1, -7, -13, -19, \ldots$; with $d = -6$

Example 1 Writing an Arithmetic Sequence

Write the first 5 terms of:

a) an increasing arithmetic sequence 17, 22, 27, 32, 37, 42 (examples)

b) a decreasing arithmetic sequence 13, 11, 9, 7, 5

Consider this arithmetic sequence: 3, 7, 11, 15, 19, 23, . . .
To determine an expression for the **general term,** t_n, use the pattern in the terms. The common difference is 4. The first term is 3.

t_1	$3 + 4(0)$
t_2	$3 + 4(1)$
t_3	$3 + 4(2)$
t_4	$3 + 4(3)$
⋮	
t_n	$3 + 4(n-1)$

For each term, the second factor in the product is 1 less than the term number.

The second factor in the product is 1 less than n, or $n - 1$.

Write: $t_n = 3 + 4(n-1)$

↑ general term ↑ first term ↑ common difference

The General Term of an Arithmetic Sequence

An arithmetic sequence with first term, t_1, and common difference, d, is: $t_1, t_1 + d, t_1 + 2d, t_1 + 3d, \ldots$

The general term of this sequence is: $t_n = t_1 + d(n - 1)$

$t_1 = t_1$
$t_2 = t_1 + d$
$t_3 = t_1 + 2d$
$t_4 = t_1 + 3d$
$t_5 = t_1 + 4d$
$t_n = t_1 + d(n-1)$

Example 2 **Calculating Terms in a Given Arithmetic Sequence**

For this arithmetic sequence: −3, 2, 7, 12, . . .

a) Determine t_{20}.

$t_n = t_1 + d(n-1)$
$t_{20} = -3 + 5(19)$
$= -3 + 95$
$= 92$

b) Which term in the sequence has the value 212?

$t_n = t_1 + d(n-1)$
$212 = -3 + 5(n-1)$
$212 = -3 + 5n - 5$
$220 = 5n$
$44 = n$

$t_{44} = 212$

Example 3 Calculating a Term in an Arithmetic Sequence, Given Two Terms

Two terms in an arithmetic sequence are $t_3 = 4$ and $t_8 = 34$.
What is t_1?

$$\underset{1}{\_\_} \quad \underset{2}{\_\_} \quad \underset{3}{\underline{4}} \quad \underset{4}{\_\_} \quad \underset{5}{\_\_} \quad \underset{6}{\_\_} \quad \underset{7}{\_\_} \quad \underset{8}{\underline{34}}$$

$+d \quad +d \quad +d \quad +d \quad +d$

$t_8 = t_3 + 5d$
$34 = 4 + 5d$
$30 = 5d$
$6 = d$

$t_1 = t_3 - 2d$
$t_1 = 4 - 2(6)$
$t_1 = 4 - 12$
$t_1 = -8$

Example 4 Using an Arithmetic Sequence to Model and Solve a Problem

Some comets are called periodic comets because they appear regularly in our solar system. The comet Kojima appears about every 7 years and was last seen in the year 2007. Halley's comet appears about every 76 years and was last seen in 1986.
Determine whether both comets should appear in 3043.

$t_1 = 2007$
$d = 7$

$t_n = t_1 + d(n-1)$
$t_n = 2007 + 7(n-1)$
$3043 = 2007 + 7n - 7$
$3043 = 2000 + 7n$
$1043 = 7n$
$149 = n$

Kojima should appear in 3043

$t_1 = 1986$
$d = 76$

$t_n = 1986 + 76(n-1)$
$3043 = 1986 + 76n - 76$
$3043 = 1910 + 76n$
$1133 = 76n$
$14.9078 = n$

Since n is not an even number then no, Haley comet will not appear that year.

Answer Key

Topic 1.2 Arithmetic Series

Carl Friedrich Gauss was a mathematician born in Braunschweig, Germany, in 1777. He is noted for his significant contributions in fields such as number theory, statistics, astronomy, and differential geometry. When Gauss was 10, his mathematics teacher challenged the class to find the sum of the numbers from 1 to 100. Believing that this task would take some time, the teacher was astounded when Gauss responded with the correct answer of 5050 within minutes.

Gauss used a faster method than adding each individual term.

First, he wrote the sum twice, once in ascending order and the other in a descending order. Gauss then took the sum of the two rows.

$$\begin{array}{l} 1 + 2 + 3 + 4 + \cdots + 99 + 100 \\ \underline{100 + 99 + \cdots + 4 + 3 + 2 + 1} \\ 101 + 101 + \cdots + 101 + 101 \end{array}$$

What do you think Gauss did next?

$$S = 1 + 2 + 3 + \ldots + 99 + 100$$
$$\underline{S = 100 + 99 + 98 + \ldots + 2 + 1}$$
$$2S = 101 + 101 + 101 + \ldots + 101 + 101$$

101 is the sum of the firt term + Last term
$101 = t_1 + t_n$

$$2S = 100(101)$$
$$\frac{2S}{2} = \frac{100(t_1 + t_n)}{2}$$

$n = \#$ terms
$n = 100$

$$S = \frac{n}{2}(t_1 + t_n)$$

formula

In determining the sum of the numbers from 1 to 100, Gauss had discovered the underlying principles of an **arithmetic series**.

S_n represents the sum of the first n terms of a series.

S_n is read as "S subscript n" or "S sub n."

In the series $2 + 4 + 6 + 8 + \cdots$, S_4 is the sum of the first four terms.

You can use Gauss's method to derive a formula for the sum of the general arithmetic series.

The general arithmetic series may be written as
$t_1 + (t_1 + d) + (t_1 + 2d) + \cdots + [(t_1 + (n - 3)d] + [(t_1 + (n - 2)d]$
$+ [(t_1 + (n - 1)d]$

For this series, t_1 is the first term
n is the number of terms
d is the common difference

Use Gauss's method.
Write the series twice, once in ascending order and the other in descending order. Then, sum the two series.

$$\begin{array}{rcl} S_n &=& t_1 + (t_1 + d) + \cdots + [t_1 + (n-2)d] + [t_1 + (n-1)d] \\ S_n &=& [t_1 + (n-1)d] + [t_1 + (n-2)d] + \cdots + (t_1 + d) + t_1 \\ \hline 2S_n &=& [2t_1 + (n-1)d] + [2t_1 + (n-1)d] + \cdots + [2t_1 + (n-1)d] + [2t_1 + (n-1)d] \\ 2S_n &=& n[2t_1 + (n-1)d] \\ S_n &=& \frac{n}{2}[2t_1 + (n-1)d] \end{array}$$

The *sum* of an arithmetic series can be determined using the formula
$$S_n = \frac{n}{2}[2t_1 + (n - 1)d]$$
where t_1 is the first term
n is the number of terms
d is the common difference
S_n is the sum of the first n terms

A variation of this general formula can be derived by substituting t_n for the formula for the general term of an arithmetic sequence.

$$S_n = \frac{n}{2}[2t_1 + (n - 1)d]$$

$$S_n = \frac{n}{2}[t_1 + t_1 + (n - 1)d] \quad \text{Since } t_n = t_1 + (n-1)d.$$

$$S_n = \frac{n}{2}(t_1 + t_n)$$

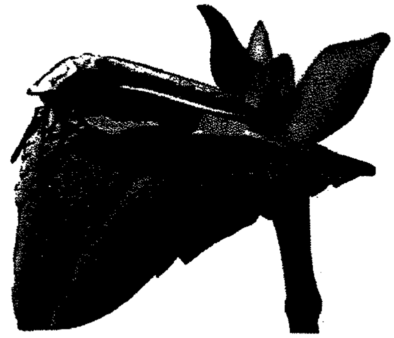
The *sum* of an arithmetic series can be determined using the formula
$$S_n = \frac{n}{2}(t_1 + t_n)$$
where t_1 is the first term
n is the number of terms
t_n is the nth term
S_n is the sum of the first n terms

How would you need to express the last terms of the general arithmetic series in order to directly derive this formula using Gauss's method?

Example 1

Determine the Sum of an Arithmetic Series

Male fireflies flash in various patterns to signal location or to ward off predators. Different species of fireflies have different flash characteristics, such as the intensity of the flash, the rate of the flash, and the shape of the flash. Suppose that under certain circumstances, a particular firefly flashes twice in the first minute, four times in the second minute, and six times in the third minute.



a) If this pattern continues, what is the number of flashes in the 30th minute?

b) What is the total number of flashes in 30 min?

a) $2, 4, 6, 8, \ldots, t_{30}$

1st term

30th term

$t_{30} = t_1 + d(n-1)$

$t_{30} = 2 + 2(30-1)$

$t_{30} = 2 + 2(29)$

$t_{30} = 60$

$S_{30} = \frac{30}{2}(2+60)$

$S_{30} = 15(62) = 930$

Your Turn

Determine the total number of flashes for the male firefly in 42 min.

$2, 4, 6, \ldots,$

1st 2nd 3rd 42nd

$2 + 4 + 6 + \ldots + 84$

$t_n = t_1 + d(n-1)$

$t_n = 2 + 2(41)$

$t_n = 84$

$S_n = \frac{n}{2}(t_1 + t_n)$

$S_{42} = 21(2+84)$

$S_{42} = 21(86) = 1806$

Example 2

Determine the Terms of an Arithmetic Series

The sum of the first two terms of an arithmetic series is 13 and the sum of the first four terms is 46. Determine the first six terms of the series and the sum to six terms.

$t_1 + t_1 + d = 13 \quad t_1 + t_1 + d + t_1 + 2d + t_1 + 3d = 46$

$2t_1 + d = 13$
$4t_1 + 6d = 46$
$4t_1 + 2d = 26$
$4d = 20$
$d = 5$

$2t_1 + 5 = 13$
$2t_1 = 8$
$t_1 = 4$

4, 9, 14, 19, 24, 29

$S_6 = \frac{6}{2}(4 + 29)$
$S_6 = 3(33) = 99$

Your Turn

The sum of the first two terms of an arithmetic series is 19 and the sum of the first four terms is 50. What are the first six terms of the series and the sum to 20 terms?

$t_1 + t_1 + d = 19$
$t_1 + t_1 + d + t_1 + 2d + t_1 + 3d = 50$

$2t_1 + d = 19$
$4t_1 + 6d = 50$

$4t + 2d = 38$
$4t + 6d = 50$
$-4d = -12$
$d = 3$

$2t_1 + 3 = 19$
$2t_1 = 16$
$t_1 = 8$

8, 11, 14, 17, 20, 23

$t_{20} = t_1 + d(n-1)$
$= 8 + 3(19)$
$= 8 + 57$
$= 65$

$S_{20} = 10(8 + 65) = 10(31) = 310$
$= 10(73) = 730$

Key Ideas

- Given the sequence $t_1, t_2, t_3, t_4, \ldots, t_n$ the associated series is $S_n = t_1 + t_2 + t_3 + t_4 + \cdots + t_n$.
- For the general arithmetic series,
  $t_1 + (t_1 + d) + (t_1 + 2d) + \cdots + (t_1 + [n - 1]d)$ or
  $t_1 + (t_1 + d) + (t_1 + 2d) + \cdots + (t_n - d) + t_n$,
  the sum of the first n terms is
  $S_n = \frac{n}{2}[2t_1 + (n - 1)d]$ or $S_n = \frac{n}{2}(t_1 + t_n)$,
  where t_1 is the first term
  n is number of terms
  d is the common difference
  t_n is the nth term
  S_n is the sum to n terms

$t_n = t_1 + d(n-1)$

Answer Key

Assess Your Understanding

1.1

$\underline{27}$ $\underline{19}$ $\underline{11}$ $\underline{3}$ $\underset{t_5}{\underline{-5}}$ $\underline{-13}$ $\underline{-21}$ $\underline{-29}$ $\underline{-37}$ $\underset{t_{10}}{\underline{-45}}$

1. Multiple Choice Which arithmetic sequence has $d = -8$ and $t_{10} = -45$?

A. 27, 19, 11, 3, ... B. −8, −12, −16, −20, ...
C. −5, −13, −21, −29, ... D. −27, −19, −11, −3, ...

2. Write the first 4 terms of an arithmetic sequence with its 5th term equal to −4.

$\underline{-12}$ $\underline{-10}$ $\underline{-8}$ $\underline{-6}$ $\underline{-4}$ $\underline{-2}$

3. This sequence is arithmetic: −8, −11, −14, ...

a) Write a rule for the nth term.

$t_n = t_1 + d(n-1)$
$t_n = -8 - 3(n-1)$
$t_n = -8 - 3n + 3$

$t_n = -3n - 5$

b) Use your rule to determine the 17th term.

$t_{17} = -3(17) - 5$
$= -51 - 5$
$= -56$

4. Use the given data about each arithmetic sequence to determine the indicated values.

a) $t_4 = -5$ and $t_7 = -20$; determine d and t_1

$\underline{10}$ $\underline{5}$ $\underline{0}$ $\underset{t_4}{\underline{-5}}$ $\underline{-10}$ $\underline{-15}$ $\underset{t_7}{\underline{-20}}$

$t_1 = 10$

$t_4 + 3d = t_7$
$-5 + 3d = -20$
$3d = -15$
$d = -5$

b) $t_1 = 3, d = 4$, and $t_n = 59$; determine n

$t_n = t_1 + d(n-1)$

$59 = 3 + 4(n-1)$

$59 = 3 + 4n - 4$

$59 = 4n - 1$

$60 = 4n$

$15 = n$

5. The steam clock in the Gastown district of Vancouver, B.C., displays the time on four faces and announces the quarter hours with a whistle chime that plays the tune *Westminster Quarters.* This sequence represents the number of tunes played from 1 to 3 days: 96, 192, 288, . . . Determine the number of tunes played in one year.

$d = 96$

$t_n = ?$

$\underline{96}\ \ \underline{192}$

$t_1\ \ t_2$

$t_n = t_1 + d(n-1)$

$t_{365} = 96 + 96(364)$

$= 35\,040$

1.2

6. Multiple Choice For which series could you use $S_n = \frac{n(t_1 + t_n)}{2}$ to determine its sum?

A. $3 + 5 + 7 + 10 + 13 + 17 + 21$

B. $3 - 1 - 5 - 9 - 13 - 17 - 21$

C. $-3 - 5 - 8 - 10 - 13 - 15 - 18$

D. $3 - 1 + 5 - 3 + 7 - 5 + 9$

$d = -4$

7. a) Create the first 5 terms of an arithmetic series with a common difference of −3.

$\underline{-3}\ \ \underline{-6}\ \ \underline{-9}\ \ \underline{-12}\ \ \underline{-15}$

b) Determine S_{26} for your series.

$S_{26} = \frac{\cancel{26}^{13}(2(-3) + (-3)(25))}{\cancel{2}_1}$

$= 13(-6 + -75)$

$= 13(-81) = -1053$

Step 1 → Find n

$t_n = t_1 + d(n-1)$
$158 = -2 + 5(n-1)$
$158 = -2 + 5n - 5$
$158 = 5n - 7$
$165 = 5n$
$33 = n$

8. Determine the sum of this arithmetic series:

$-2 + 3 + 8 + 13 + \ldots + 158$

$t_1 = -2$
$d = 5$
$t_n = 158$

step 2 use $S_n = \frac{n(t_1 + t_n)}{2}$

$$S_{33} = \frac{33(-2+158)}{2} = \frac{33(156)}{2} = 2574$$

9. Use the given data about each arithmetic series to determine the indicated value.

a) $S_{17} = 106.25$ and $t_{17} = 8.25$; determine t_1

$S_n = \frac{n(t_1 + t_n)}{2}$

$106.25 = \frac{17(t_1 + 8.25)}{2}$

$106.25 = \frac{17t_1 + 140.25}{2}$

$212.5 = 17t_1 + 140.25$

$\frac{72.25}{17} = \frac{17t_1}{17}$

$4.25 = t_1$

b) $S_{15} = 337.5$ and $t_1 = -2$; determine d

Use $S_n = \frac{n(t_1 + t_n)}{2}$ to find t_{15}

$337.5 = \frac{15(-2 + t_{15})}{2}$

$337.5 = \frac{-30 + 15t_{15}}{2}$

$675 = -30 + 15t_{15}$

$705 = 15t_{15}$

$47 = t_{15}$

now use $t_n = t_1 + d(n-1)$

$t_n = 47$
$t_1 = -2$

$47 = -2 + d(15-1)$
$47 = -2 + 14d$
$49 = 14d$
$3.5 = d$

Answer Key

Math 20 – 1 Pre-Calculus 11 Topic 1.3 Geometic Sequence

FOCUS **Solve problems involving geometric sequences.**

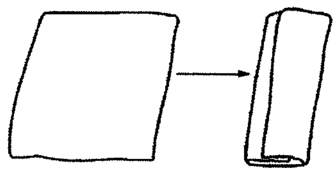
Get Started

For each sequence below, what are the next 2 terms? What is the rule?

- $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ $\frac{1}{6}, \frac{1}{7}$ add 1 to the denominators
- $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$ $\frac{1}{243}$ $\frac{1}{729}$ denom. are powers of 3
- $1, -3, 9, -27, 81, \ldots$ $-243, 729$ × by -3

Construct Understanding

A French pastry called *mille feuille* or "thousand layers" is made from dough rolled into a square, buttered, and then folded into thirds to make three layers. This process is repeated many times. Each step of folding and rolling is called a turn.



How many turns are required to get more than 1000 layers? → 7

No. of turns	No. of Layers
1	3
2	9
3	27
4	81
5	243
6	729
7	2187

A **geometric sequence** is formed by multiplying each term after the 1st term by a constant, to determine the next term.

For example, $4, 4(3), 4(3)^2, 4(3)^3, \ldots$, is the geometric sequence: $4, 12, 36, 108, \ldots$

The first term, t_1, is 4 and the constant is 3.

The constant is the common ratio, r, of any term after the first, to the preceding term.

$$r = \frac{t_2}{t_1}$$

The common ratio is any non-zero real number.

To determine the common ratio, divide any term by the preceding term.

For the geometric sequence above:

$r = \frac{12}{4}$ and $r = \frac{36}{12}$ and $r = \frac{108}{36}$

$r = 3$ $\quad r = 3$ $\quad r = 3$

The sequence $4, 12, 36, 108, \ldots$, is an **infinite geometric sequence** because it continues forever.

The sequence 4, 12, 36, 108 is a **finite geometric sequence** because the sequence is limited to a fixed number of terms.

Here are some other examples of geometric sequences.

- This is an *increasing* geometric sequence because the terms are increasing: 2, 10, 50, 250, 1250, . . .
  The sequence is **divergent** because the terms do not approach a constant value.
- This is a geometric sequence that neither increases, nor decreases because consecutive terms have numerically greater values and different signs: 1, −2, 4, −8, 16, . . .
  The sequence is *divergent* because the terms do not approach a constant value.
- This is a *decreasing* geometric sequence because the terms are decreasing:
  $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$
  The sequence is **convergent** because the terms approach a constant value of 0.

Example 1 Determining a Term of a Given Geometric Sequence

a) Determine the 12th term of this geometric sequence: 512, −256, 128, −64, . . .
b) Identify the sequence as convergent or divergent.

$r = \frac{-256}{512} = -\frac{1}{2}$

$t_1 = 512$
$t_2 = 512\left(-\frac{1}{2}\right)$
$t_3 = 512\left(-\frac{1}{2}\right)^2$
$t_4 = 512\left(-\frac{1}{2}\right)^3$
$\vdots$
$t_{12} = 512\left(-\frac{1}{2}\right)^{11} = -\frac{1}{4}$

and since consecutive terms approach zero, the geometric sequence is convergent.

Example 2 Creating a Geometric Sequence

Create a geometric sequence whose 5th term is 48.

__ __ __ __ 48
t_5

if $r = 2$ then divide 48 by 2 to get t_4

__ __ __ 24 48

keep going

3 6 12 24 48

A geometric sequence with first term, t_1, and common ratio, r, can be written as:

$$t_1,\ t_1r,\ t_1r^2,\ t_1r^3,\ t_1r^4,\ \ldots,\ t_1r^{n-1}$$
$$\uparrow\ \ \uparrow\ \ \uparrow\ \ \uparrow\ \ \uparrow\ \ \qquad \uparrow$$
$$t_1\ \ t_2\ \ t_3\ \ t_4\ \ t_5\ \ \qquad t_n$$

The exponent of each power of the common ratio is 1 less than the term number.

The General Term of a Geometric Sequence

For a geometric sequence with first term, t_1, and common ratio, r, the general term, t_n, is:

$t_n = t_1r^{n-1}$

Recall that the product of two negative numbers is positive. So, a square number may be the product of two equal negative numbers or two equal positive numbers. For example,

when $r^2 = 4$

then $r = \pm\sqrt{4}$

and $r = 2$ or $r = -2$

To determine the common ratio of a geometric sequence, you may need to solve an equation of this form:

$r^4 = 81$

then $r^2 = 9$

and $r = 3$ or $r = -3$

Example 3 Determining Terms and the Number of Terms in a Finite Geometric Sequence

In a finite geometric sequence, $t_1 = 5$ and $t_5 = 1280$

a) Determine t_2 and t_6.

b) The last term of the sequence is 20 480. How many terms are in the sequence?

(a) $t_n = t_1 r^{n-1}$

$1280 = 5r^{5-1}$

$1280 = 5r^4$

$256 = r^4$

$\pm\sqrt[4]{256} = r$

$\pm 4 = r$

$t_1 = 5$

$t_2 = 5(4) = 20$ or $5(-4) = -20$

$t_6 = 5(-4)^5$ or $5(4)^5$

$= -5120$ or 5120

Example 4 Using a Geometric Sequence to Solve a Problem

The population of Airdrie, Alberta, experienced an average annual growth rate of about 9% from 2001 to 2006. The population in 2006 was 28 927. Estimate the population in each year to the nearest thousand.

a) 2011

b) 2030, the 125th anniversary of Alberta becoming part of Canada

What assumption did you make? Is this assumption reasonable?

$\hookrightarrow$ on back

(b) $t_n = t_1 r^{n-1}$

$\dfrac{20480}{5} = \dfrac{5r^{n-1}}{5}$

$4096 = 4^{n-1}$

guess + test to find $4^6 = 4096$

Therefore $4096 = 4^{7-1}$

so $n = 7$

There are 7 terms in the sequence.

Example 4

A growth rate of 9% means each year the pop. increases by 9% or 0.09

Pop in 2006 was 28927

2007 was $28927 + 0.09(28927)$

or $1.09(28927)$ or $28927(1.09)$

Therefore 1.09 is the common ratio and the population in 2011 $= 28927(1.09)^5$

$= 44\ 507.8$

b) To predict the population in 2030, determine n the number of years from 2006 to 2030

$n = 2030 - 2006 = 24$

Therefore pop. $= 28927(1.09)^{24} = 228\ 843.9$

Reasonable? – No trends like 9% growth rate change over time.

Math 20-1 Lesson 1.3 Geometric Sequences – Guided Practice

Name Answer Key

Check Your Understanding

1. a) Determine the 10th term of this geometric sequence: 2, –6, 18, –54, . . .

b) Identify the sequence as convergent or divergent.

$t_1 = 2$

$r = -3$

$t_1 = 2$

$t_2 = 2(-3)$

$t_{10} = 2(-3)^9 = -39\ 366$

---

2. Create a geometric sequence whose 6th term is 27.

864 432 216 108 54 27 ___

6th

$r = \frac{1}{2}$

---

3. In a finite geometric sequence, $t_1 = 7$ and $t_5 = 567$

a) Determine t_2 and t_6.

b) The last term is 45 927. How many terms are in the sequence?

7 __ __ __ 567

$t_n = t_1 r^{n-1}$

$567 = 7r^4$

$81 = r^4$

$\pm 3 = r$

$t_2 = 7(3) = 21$ or -21

$t_6 = 7(3)^5$ or $7(-3)^5$

1701 or –1701

$t_n = t_1 r^{n-1}$

$45927 = 7(3)^{n-1}$

$6561 = 3^{n-1}$

since $3^8 = 6561$

$n = 9$

---

4. Statistics Canada estimates the population growth of Canadian cities, provinces, and territories. The population of Nunavut is expected to grow annually by 0.8%. In 2009, its population was about 30 000. Estimate the population in each year to the nearest thousand.

a) 2013

b) 2049; Nunavut's 50th birthday

1.4 Geometric Series

The Sum of *n* Terms of a Geometric Series

For the geometric series $t_1 + t_1r + t_1r^2 + \ldots + t_1r^{n-1}$,

the sum of n terms, S_n, is: $S_n = \dfrac{t_1(1 - r^n)}{1 - r}, r \neq 1$

1. Determine the sum of the first 12 terms of this geometric series:
$3 + 12 + 48 + 192 + \ldots$

$t_1 = 3$ and r is: $\frac{12}{3} = 4$

Use: $S_n = \dfrac{t_1(1 - r^n)}{1 - r}, r \neq 1$

Substitute: $n = 12, t_1 = 3, r = 4$

$S_{12} = \dfrac{3(1 - 4^{12})}{1 - 4}$

$S_{12} = 16\ 777\ 215$

The sum of the first 12 terms is 16 777 215.

2. The sum of the first 14 terms of a geometric series is 16 383. The common ratio is -2. Determine the 1st term.

Use: $S_n = \dfrac{t_1(1 - r^n)}{1 - r}, r \neq 1$

Substitute: $n = 14, S_n = 16\ 383, r = -2$

$16\ 383 = \dfrac{t_1(1 - (-2)^{14})}{1 - (-2)}$

$16\ 383 = \dfrac{t_1(1 - 16\ 384)}{3}$

$16\ 383 = -5461t_1$

$\dfrac{16\ 383}{-5461} = t_1$

$-3 = t_1$

The 1st term is -3.

3. Calculate the sum of this geometric series:
$-3 - 15 - 75 - \ldots - 46\,875$

$t_1 = -3$ and r is $\frac{-15}{-3} = 5$
To determine n, use: $t_n = t_1 r^{n-1}$
Substitute: $t_n = -46\,875$, $t_1 = -3, r = 5$
$-46\,875 = (-3)(5)^{n-1}$
$15\,625 = 5^{n-1}$
$5^6 = 5^{n-1}$
$6 = n - 1$
$n = 7$
There are 7 terms in the series.
Use: $S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1$
Substitute: $n = 7, t_1 = -3$, $r = 5$
$S_7 = \frac{(-3)(1 - 5^7)}{1 - 5}$
$S_7 = -58\,593$
The sum is $-58\,593$.

4. A person takes tablets to cure a chest infection. Each tablet contains 500 mg of an antibiotic. About 15% of the mass of the antibiotic remains in the body when the next tablet is taken. Determine the mass of antibiotic in the body after each number of tablets:

a) 3 tablets b) 10 tablets

a)

Number of tablets	Mass of antibiotic (mg)
1	500
2	$500 + 500(0.15)$ $= 575$
3	$500 + 500(0.15)$ $+ 500(0.15)^2$ $= 586.25$

The sum is 586.25. So, after taking the 3rd tablet, the total mass is 586.25 mg, or about 586 mg.

b) Determine the sum of this geometric series:
$500 + 500(0.15) + 500(0.15)^2 + 500(0.15)^3 + \ldots + 500(0.15)^9$
Use: $S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1$
Substitute: $n = 10$, $t_1 = 500$, and $r = 0.15$
$S_{10} = \frac{500(1 - 0.15^{10})}{1 - 0.15}$
$S_{10} = 588.2352\ldots$
The mass of antibiotic is about 588 mg.

1.6 Infinite Geometric Series

FOCUS **Determine the sum of an infinite geometric series.**

An **infinite geometric series** has an infinite number of terms.
For an infinite geometric series, if the sequence of partial sums converges to a constant value as the number of terms increases, then the geometric series is convergent and the constant value is the finite sum of the series.
This sum is called the **sum to infinity** and is denoted by S_∞.

Write $0.\overline{6}$ as a series

$0.\overline{6} = 0.6 + 0.06 + 0.006 + \ldots$
geometric series because $r = \frac{1}{10}$

Example 1 Estimating the Sum of an Infinite Geometric Series

Predict whether each infinite geometric series has a finite sum.
Estimate each finite sum.

a) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$

b) $0.5 + 1 + 2 + 4 + \ldots$

c) $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \ldots$

a) $S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$S_3 = \frac{3}{4} + \frac{1}{8} = \frac{7}{8}$

$S_4 = \frac{7}{8} + \frac{1}{16} = \frac{15}{16}$

$S_5 = \frac{15}{16} + \frac{1}{32} = \frac{31}{32}$

The sum appears to be a finite sum of one.

b) $S_1 = 0.5$

$S_2 = 1.5$

$S_3 = 3.5$

$S_4 = 7.5$

No finite sum. (diverges)

c) $S_1 = \frac{1}{2} = 0.5$

$S_2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} = 0.25$

$S_3 = \frac{1}{4} + \frac{1}{8} = \frac{3}{8} = 0.375$

$S_4 = 0.375 - \frac{1}{16} = 0.3125$

$S_5 = 0.3125 + \frac{1}{32} = 0.34375$

$S_6 = 0.34375 - \frac{1}{64} = 0.328125$

The sum appears to be a finite sum of 0.33 (approx.)

* If the common ratio is between -1 and $+1$, then the infinite geometric series will converge.

The Sum of an Infinite Geometric Series

For an infinite geometric series with first term, t_1, and common ratio, $-1 < r < 1$, the sum of the series, S_∞, is:

$$S_\infty = \frac{t_1}{1 - r}$$

Example 2 Determining the Sum of an Infinite Geometric Series

Determine whether each infinite geometric series converges or diverges. If it converges, determine its sum.

a) $27 - 9 + 3 - 1 + \ldots$ **b)** $4 - 8 + 16 - 32 + \ldots$

a)

$$S_\infty = \frac{t_1}{1-r}$$

$$= \frac{27}{1-(-\frac{1}{3})}$$

$$= \frac{27}{1.\overline{3}}$$

$$= 20.25$$

b)

$S_\infty = ?$

Since $r = -2$ there will be no finite sum

Example 3 Using an Infinite Geometric Series to Solve a Problem

Determine a fraction that is equal to $0.4\overline{9}$.

$$0.4\overline{9} = 0.4 + \underbrace{0.09 + 0.009 + 0.0009 + \ldots}$$

$$Sum_\infty = \frac{t_1}{1-r} = \frac{0.09}{1-\frac{1}{10}} = \frac{0.09}{0.9} = \frac{1}{10}$$

$$0.4\overline{9} = 0.4 + \frac{1}{10} = \frac{5}{10} = \frac{1}{2}$$

Guided Practice Questions

1. Predict whether each infinite geometric series has a finite sum. Estimate each finite sum.

a) $\frac{1}{3} + \frac{1}{12} + \frac{1}{48} + \frac{1}{192} + \ldots \frac{1}{768}$ finite

b) $-4 - 8 - 16 - 32 - \ldots -64$ not finite

c) $\frac{1}{10} - \frac{1}{100} + \frac{1}{1000} - \frac{1}{10\,000} + \ldots \frac{1}{100\,000}$ finite

a) $S_1 = \frac{1}{3} = 0.\overline{3}$

$S_2 = 0.41\overline{6}$

$S_3 = 0.4375$

$S_4 = 0.4427$

$S_5 = 0.4440$

est. sum $= 0.\overline{4}$

b) None

c) $S_1 = \frac{1}{10}$ or 0.1

$S_2 = 0.09$

$S_3 = 0.091$

$S_4 = 0.0909$

est. sum $= 0.\overline{09}$

2. Determine whether each infinite geometric series converges or diverges. If it converges, determine its sum.

a) $32 + 8 + 2 + 0.5 + \ldots$

b) $100 - 10 + 1 - 0.1 + \ldots$

a) $t_1 = 32$ $r = \frac{8}{32} = \frac{1}{4}$

$$S_\infty = \frac{32}{1-\frac{1}{4}} = \frac{32}{0.75} = 42.\overline{6}$$

b) $t_1 = 100$ $r = -0.1$

$$S_\infty = \frac{t_1}{1-r} = \frac{100}{1--0.1} = 90.\overline{90}$$

3. Determine a fraction that is equal to $0.1\overline{6}$.

$$0.1\overline{6} = 0.1 + \underbrace{0.06 + 0.006 + 0.0006 + \ldots}_{\text{series.}}$$

$$Sum_\infty = \frac{0.06}{1-\frac{1}{10}} = \frac{0.06}{0.9} = \frac{6}{90} = \frac{2}{30}$$

$$0.1\overline{6} = \frac{1}{10} + \frac{2}{30} = \frac{3}{30} + \frac{2}{30} = \frac{5}{30} = \frac{1}{6}$$